(2008)

MATH2101 Complex Analysis
Examination questions and solutions

1. (a) i. Give the definition of the derivative of a function f at a point $z_0 \in \mathbb{C}$.

   ii. What does it mean for a function f to be holomorphic in the domain $\Omega \subset \mathbb{C}$?

   (b) Assuming that the function f is holomorphic in the disk $D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ prove that $g(z) = \overline{f(\overline{z})}$ is also holomorphic in $D(0,1)$ and find its derivative.

   (c) Find the radii of convergence of the following series, stating clearly which results you are using:

$$\sum_{k=0}^{\infty} k^{113} 2^{-k} (z-1)^k, \quad \sum_{n=2}^{\infty} n!(z-e)^{3n}, \quad \sum_{k=0}^{\infty} \frac{z^k}{(k!)^2}.$$

Solution.

(a) i. The limit

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

is called the derivative at z_0 if it exists.

ii. A function f is said to be holomorphic in a domain Ω if it is differentiable at every point of Ω.

Bookwork

(b) Clearly $z \in D(0,1)$ iff $\overline{z} \in D(0,1)$.

Write:

$$\frac{g(z+h) - g(h)}{h} = \frac{\overline{f(\overline{z} + \overline{h})} - \overline{f(\overline{z})}}{h} = \overline{\left[\frac{f(\overline{z} + \overline{h}) - f(\overline{z})}{\overline{h}}\right]}.$$

Denote $\overline{z} = w, \zeta = \overline{h}$. Then the above ratio is rewritten as follows:

$$\overline{\left[\frac{f(w+\zeta) - f(w)}{\zeta}\right]} \to \overline{f'(w)} = \overline{f'(\overline{z})}$$

as $\zeta = \overline{h} \to 0$. Conclusion: $g \in H(D(0,1))$ and $g'(z) = \overline{f'(\overline{z})}$.

Seen exercise

(c) Use the Ratio Test for all three series. For the first one

$$a_k = k^{113} 2^{-k} (z-1)^k,$$

so

$$\frac{|a_{k+1}|}{|a_k|} = \left(1 + \frac{1}{k}\right)^{113} \frac{1}{2}|z-1| \to \frac{1}{2}|z-1|,$$

as $k \to \infty$. According to the Ratio Test, the series converges absolutely if $|z-1|/2 < 1$ and diverges if $|z-1|/2 > 1$. Therefore the radius of convergence is $R = 2$.

For the second series, with $a_n = n!(z-e)^{3n}$,

$$\frac{|a_{n+1}|}{|a_n|} = (n+1)|z-e|^3 \to \infty$$

as $n \to \infty$. Thus the series diverges for all $z \neq e$. Consequently, $R = 0$.

For the third series, with $a_k = z^k (k!)^{-2}$,

$$\frac{|a_{k+1}|}{|a_k|} = \frac{(k!)^2}{((k+1)!)^2}|z| = \frac{1}{(k+1)^2}|z| \to 0,$$

as $k \to \infty$. Thus the series converges for all $z \in \mathbb{C}$, and hence $R = \infty$.

Unseen exercise

2. (a) State the Cauchy-Riemann equations for real-valued functions $u(x,y), v(x,y)$.

(b) Suppose that $f(z) = u(x,y) + iv(x,y)$ is holomorphic on a domain D, and that $|f(z)| = \text{const}$ throughout D. Show that $f = \text{const}$ as well.

(c) i. Write the MacLaurin series for $\sin z$ and $\cos z$.

ii. Using the sum formula for the cosine function, write the Taylor expansion of $\cos z$ centered at z_0 (i.e. expansion in powers of $z - z_0$), where z_0 is a point in $\mathbb{C}$.

iii. Prove that $\frac{d}{dz} \sin z = \cos z$ and $\frac{d}{dz} \cos z = -\sin z$, stating clearly which results you are using.

iv. Prove that $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$.

Solution.

(a) The Cauchy-Riemann equations are $u_x = v_y$, $u_y = -v_x$.

Bookwork

(b) Let $|f| = \sqrt{u^2 + v^2} = c$. If $c = 0$, than $f = 0$.
Suppose that $c \neq 0$. Since $|f| = c$, we have $u^2 + v^2 = c^2$, so that

$$uu_x + vv_x = 0, \quad uu_y + vv_y = 0,$$

whence, by Cauchy-Riemann equations,

$$uu_x - vu_y = 0, uu_y + vu_x = 0.$$

Multiply the first equation by u and the second by v:

$$u^2u_x - uvu_y = 0, uvu_y + v^2u_x = 0.$$

Add them up:

$$0 = (u^2 + v^2)u_x = c^2u_x,$$

so that $u_x = 0$, $u_y = 0$ (as $c \neq 0$). Similarly,

$$uvu_x - v^2u_y = 0, u^2u_y + uvu_x = 0,$$

and subtracting the first from the second we obtain:

$$0 = (u^2 + v^2)u_y = c^2u_y,$$

so that $u_y = 0$. Thus $u = const$.

From the Cauchy-Riemann equations we also get $v_x = v_y = 0$, so that $v = const$. Therefore $f = const$ as claimed.

Seen exercise

(c) i.

$$\sin z = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)!}, \quad \cos z = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2k)!}.$$

ii.

$$\begin{aligned}\cos z = & \cos(z - z_0 + z_0) = \cos(z - z_0)\cos z_0 - \sin(z - z_0)\sin z_0 \\ = & \cos z_0 \sum_{k=0}^{\infty}(-1)^k \frac{(z-z_0)^{2k}}{(2k)!} - \sin z_0 \sum_{k=0}^{\infty}(-1)^k \frac{(z-z_0)^{2k+1}}{(2k+1)!}.\end{aligned}$$

iii. By the rule of differentiation for power series, the derivative of a series is again a series with the same radius of convergence and terms obtained by term-by-term differentiation. Thus

$$\frac{d}{dz}\sin z = \sum_{k=0}^{\infty}(-1)^k \frac{d}{dz}\frac{z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2k)!} = \cos z,$$

and

$$\begin{aligned}\frac{d}{dz}\cos z = & \sum_{k=0}^{\infty}(-1)^k \frac{d}{dz}\frac{z^{2k}}{(2k)!} = \sum_{k=1}^{\infty}(-1)^k \frac{z^{2k-1}}{(2k-1)!} \\ = & \sum_{k=0}^{\infty}(-1)^{k+1}\frac{z^{2k+1}}{(2k+1)!} = -\sin z.\end{aligned}$$

iv. Let $f(z) = \sin^2 z + \cos^2 z$. Differentiate:

$$\frac{d}{dz}f(z) = 2\sin z\cos z - 2\cos z\sin z = 0.$$

This means that the function f is constant throughout the complex plane, and hence $f(z) = f(0) = 1$.

Seen exercise

3. (a) Suppose that the functions $f(z) = u(x,y) + iv(x,y)$ and $g(z) = v(x,y) + iu(x,y)$ are analytic in some domain D. Show that both u and v are constant functions.

(b) Suppose that f is continuous on the disk $D(0,R)$, $R > 0$, and holomorphic in the punctured disk $D'(z_0, R) = \{z \in \mathbb{C} : 0 < |z - z_0| < R\}$. Using Laurent's Theorem and the Estimation result, show that if f is bounded in $D'(0,R)$, then f is actually holomorphic in $D(0,R)$.

(c) Find the maximal radius $R > 0$ for which the function $f(z) = (\sin z)^{-1}$ is holomorphic in $D'(0,R)$, and find the principal part of its Laurent expansion about $z_0 = 0$.

Solution.

(a) Both pairs u, v and v, u satisfy the Cauchy-Riemann equations, so that

$$u_x = v_y, \qquad u_y = -v_x,$$
$$v_x = u_y, \qquad v_y = -u_x.$$

This implies that $u_x = v_x = u_y = v_y = 0$, so that $f' = 0, g' = 0$ throughout the domain D, so that $f = const$ and $g = const$, and hence $u = const$, $v = const$.

Unseen exercise

(b) Write the formula for the coefficients in the Laurent expansion:

$$c_k = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(\zeta)}{\zeta^{k+1}} d\zeta, \quad k \in \mathbb{Z}.$$

where Γ_r is a positively oriented circular contour centered at $z_0 = 0$ of radius $r \in (0, R)$. Let $|f(z)| \leq M$ for all $z \in D(0,R)$. By the Estimation Result

$$|c_k| \leq \frac{1}{2\pi} \max_{z \in \Gamma_r} \left| \frac{f(z)}{z^{k+1}} \right| L(\Gamma_r) \leq M r^{-k}.$$

Since r is arbitrary, for $k < 0$ we conclude that $c_k = 0$, and hence the Laurent expansion contains only the terms with non-negative powers of z, which implies that the function is holomorphic, as required.

Bookwork + unseen

(c) The roots of $\sin z$, closest to 0, are $-\pi$ and π. Therefore $R = \pi$. Using the standard Taylor series for $\sin z$, we get

$$\frac{1}{\sin z} = \frac{1}{z - \frac{z^3}{6} + \dots} = \frac{1}{z} + \frac{z}{6} + \dots.$$

The principal part of the Laurent expansion is z^{-1}.

Unseen

4. (a) State the Cauchy integral formula.

(b) Let $\Gamma_1 = \{z \in \mathbb{C} : |z - 3i| = 2\}$ and $\Gamma_2 = \{z \in \mathbb{C} : |z| = 5\}$. be two positively oriented circular contours. Evaluate the integrals

$$I_1 = \int_{\Gamma_1} \frac{z}{z^2+4} dz, \qquad I_2 = \int_{\Gamma_2} \frac{z}{z^2+4} dz,$$

clearly stating what results you are using.

(c) Using the Cauchy-Riemann equations or otherwise prove that the function $h(z) = \sin(\mathrm{Im} z)$ is not differentiable in the strip $\{z : -\pi/2 < \mathrm{Im}\, z < \pi/2\}$.

Solution.

(a) Suppose that f is holomorphic in a simply connected domain U. Let Γ be a simple closed contour contained in U, enclosing a point $z_0 \in U$. Then

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} dz.$$

Cauchy's integral formula.

Bookwork

(b) Rewrite the integrand:

$$\frac{z}{(z-2i)(z+2i)}.$$

Contour Γ_1 encloses only the pole at $2i$, whereas Γ_2 encloses both $2i$ and $-2i$. Therefore by the Cauchy integral formula

$$I_1 = 2\pi i \left.\frac{z}{z+2i}\right|_{z=2i} = \pi i.$$

The second integral is found by the Cauchy residue theorem:

$$I_2 = 2\pi i \mathrm{Res}(f, 2i) + 2\pi i \mathrm{Res}(f, -2i) = 2\pi i \left.\frac{z}{z+2i}\right|_{z=2i} + 2\pi i \left.\frac{z}{z-2i}\right|_{z=-2i} = 2\pi i.$$

Seen similar exercise

(c) Write: $h(z) = u(x,y) + iv$ with $u(x,y) = \sin y$, $v = 0$. The Cauchy Riemann equations are $u_x = v_y$, $u_y = -v_x$. Since $v = 0$, the second equation takes the form $\cos y = 0$. In the indicated strip this equation has no solutions. Therefore the Cauchy-Riemann equations are not satisfied and hence the function is not differentiable.

Seen similar exercise

5. (a) Describe three types of isolated singularities of a function f by explaining how they are related to the principal part of its Laurent expansion.

(b) Find the Laurent expansions of the function

$$f(z) = \frac{z}{z^2 - 1}$$

valid for

i. $0 < |z - 1| < 2$,

ii. $|z + 1| > 2$,

iii. $|z| > 1$.

Solution.

(a) If a function f has an isolated singularity at a point z_0, then it has the Laurent expansion of the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n.$$

If $c_k = 0$ for $k < -M, M > 0$ and $c_{-M} \neq 0$, then the function is said to have a pole of order M.

If there is no such number $N \in \mathbb{Z}$ that $c_k = 0$ for all $k < N$, then the singularity is said to be essential.

If $c_k = 0$ for all $k < 0$, then the singularity is said to be removable.

Bookwork

(b) Factorise and expand using partial fractions:

$$\frac{z}{z^2 - 1} = \frac{1}{2(z-1)} + \frac{1}{2(z+1)} = h(z) + g(z).$$

For $0 < |z - 1| < 2$ we write, using the geometric series,

$$g(z) = \frac{1}{4(2^{-1}(z-1) + 1)} = \frac{1}{4} \sum_{k=0} (-1)^k 2^{-k} (z-1)^k$$

$$= \frac{1}{4} + \sum_{k=1} (-1)^k 2^{-k-2} (z-1)^k.$$

Therefore

$$f(z) = \frac{1}{2(z-1)} + \frac{1}{4} + \sum_{k=1} (-1)^k 2^{-k-2} (z-1)^k, \quad 0 < |z-1| < 2.$$

For $|z+1| > 2$, we write:

$$h(z) = \frac{1}{2(1+z)} \frac{1}{1 - 2(z+1)^{-1}} = \frac{1}{2} \sum_{k=0} \frac{2^k}{(1+z)^{k+1}}$$

$$= \frac{1}{2(1+z)} + \sum_{k=1} \frac{2^{k-1}}{(1+z)^{k+1}}.$$

Therefore

$$f(z) = \frac{1}{z+1} + \sum_{k=1} \frac{2^{k-1}}{(z+1)^{k+1}}, \quad |z+1| > 2.$$

For $|z| > 1$, we write

$$f(z) = \frac{z}{z^2(1 - z^{-2})} = \frac{1}{z} \sum_{k=0}^{\infty} z^{-2k} = \sum_{k=0}^{\infty} z^{-2k-1}.$$

Unseen exercise

6. (a) Give the definition of the residue $\mathrm{Res}(f, z_0)$ of a function f at the point z_0.
   (b) Find $\mathrm{Res}(g, 0)$ for $g(z) = z^{-2}\cosh z$.
   (c) Evaluate the following integral by integrating around a suitable closed contour:

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{x^2+4}\,dx.$$

Solution.

(a) Let p be an isolated singularity of f, and let

$$f(z) = \sum_{-\infty}^{\infty} c_k (z-p)^k$$

be the Laurent expansion of f about p. Then $\mathrm{Res}(f, p) = c_{-1}$.

Bookwork

(b) Write the Laurent expansion for g:

$$g(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} \frac{z^{2(k-1)}}{(2k)!}.$$

The coefficient c_{-1} equals zero, so that $\mathrm{Res}(g, 0) = 0$.

Unseen exercise

(c) We consider $g(z) = \frac{e^{i3z}}{z^2+4}$ with the contour

$$\Gamma = \Gamma_0 \cup \Gamma_R,$$
$$\Gamma_0 = \{\mathrm{Im}\, z = 0,\ |\,\mathrm{Re}\, z| \le R\},\quad \Gamma_R = \{z = Re^{i\theta},\ \theta \in [0, \pi]\}.$$

Then g has simple poles at $\pm 2i$. Only $2i$ is inside Γ, and so we only have to work out the residue there:

$$\mathrm{Res}(g, 2i) = \lim_{z\to 2i} (z-2i)\frac{e^{i3z}}{z^2+4} = \lim_{z\to 2i} \frac{e^{i3z}}{z+2i} = \frac{e^{-6}}{4i}.$$

Therefore, by the Residue Theorem

$$\int_{\Gamma} g(z)dz = 2\pi i \frac{e^{-6}}{4i} = \frac{\pi e^{-6}}{2}.$$

Now we apply Jordan's Lemma with $g(z) = e^{i3z} f(z)$ where $f(z) = \frac{1}{z^2+4}$ and $\alpha = 3$. As $|f(z)| \leq \frac{1}{R^2-4}$ on Γ_R we have $M(R) = \frac{1}{R^2-4}$ and we certainly have $\lim_{R\to\infty} M(R) = 0$ and so Jordan's Lemma gives

$$\lim_{R\to\infty} \int_{\Gamma_R} g(z)dz = 0.$$

Thus we conclude that

$$\lim_{R\to\infty} \int_{\Gamma} g(z)dz = \int_{-\infty}^{\infty} \frac{e^{3ix}}{x^2+4}dx,$$

and so by considering real parts we have that

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{x^2+4}dx = \frac{\pi e^{-6}}{2}.$$

Seen similar exercise